5204: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let $f : \Re \to \Re$ be a non-constant function such that,

$$f(x+y) = \frac{f(x)+f(y)}{1+f(x)f(y)}$$

for all $x, y \in \Re$. Show that $-1 < f(x) < 1$ for all $x \in \Re$.

Solution 3 by Arkady Alt, San Jose, CA

First note that $f(x) \cdot\ f(y) \neq -1$ for any $x, y \in R$.

Since $f(x) = f\left(\frac{x}{2}+\frac{x}{2}\right) = \dfrac{2f\left(\frac{x}{2}\right)}{1+\ f^2\left(\frac{x}{2}\right)} \Rightarrow |f(x)| = \dfrac{2\left|f\left(\frac{x}{2}\right)\right|}{1+\left|\ f\left(\frac{x}{2}\right)\right|^2}$

and then we have $\left(\left|f\left(\frac{x}{2}\right)\right|-1\right)^2 \geq 0 \iff \dfrac{2\left|f\left(\frac{x}{2}\right)\right|}{1+\left|\ f\left(\frac{x}{2}\right)\right|^2} \leq 1 \iff |f(x)| \leq 1.$

If we suppose $|f(x_0)| = 1,$ for some $x_0,$ then $\left|f\left(\frac{x_0}{2}\right)\right| = 1$ and $f(x)$ becomes a constant

function. Indeed, if $f(x_0) = 1,$ then for any $x \in R$ we have $f(x+x_0) = \dfrac{f(x)+\ 1}{1+\ f(x)} = 1,$

because $f(x) = f(x) \cdot f(x_0) \neq -1.$

If $f(x_0) = -1,$ then for any $x \in R$ we have $f(x+x_0) = \dfrac{f(x)-\ 1}{1-\ f(x)} = -1,$

because $-f(x) = f(x) \cdot f(x_0) \neq -1$. Thus, $|f(x)| < 1 \iff -1 < f(x) < 1$ for any x.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo TX; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, Sydney Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Kee-Wai Lau, Hong Kong, China; David Manes, Oneonta, NY; Adrian Naco, Polytechnic University, Tirana, Albania; Paolo Perfetti, Department of Mathematics, University "Tor Vergata Roma," Italy; Boris Rays, Brooklyn, NY; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Statesboro, GA; Titu Zvonaru, Comănesti, Romania and Neculai Stanciu, Buzău, Romania, and the proposer.